Input-to-state stability

3

This chapter presents the concept of the input-to-state stability (ISS) for control systems with external input, Lyapunov-based approaches to verify the ISS property of a single system and interconnections of a large scale and, finally, the application of the ISS to feedback controller design.

3.1 Introduction to the ISS

Systems with inputs. Concept of solution

Consider the system of ordinary differential equations with external input

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{3.1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ and $\boldsymbol{u}(t) \in \mathbb{R}^m$ denote the state and the input at time $t \geq 0$, respectively, function $\boldsymbol{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz with $\boldsymbol{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$, input $\boldsymbol{u} \in \mathscr{U} = L_\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ – the space of Lebesgue measurable essentially bounded functions equipped with the norm

$$\|\boldsymbol{u}\|_\infty := \operatorname*{ess\,sup}_{t \geq 0} |\boldsymbol{u}(t)| = \inf_{D \subset \mathbb{R}, \mu(D) = 0} \sup_{t \in \mathbb{R}_{\geq 0} \setminus D} |\boldsymbol{u}(t)|, \tag{3.2}$$

where $|\cdot|$ stands for the Euclidean norm.

Definition 3.1

A function $\phi : [a, b] \to \mathbb{R}$ is called *absolutely continuous* on $[a, b]$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite r and any pairwise disjoint sub-intervals (a_k, b_k) of $[a, b]$ with $\sum_{k=1}^{r} |b_k - a_k| < \delta$ it follows that $\sum_{k=1}^{r} |\phi(b_k) - \phi(a_k)| < \varepsilon$.

Example 3.1. *(a) The function*

$$\phi(x) = \begin{cases} 0, & \text{if} \quad x = 0, \\ x \sin \frac{1}{x}, & \text{if} \quad x \neq 0 \end{cases}$$

is uniformly continuous but not absolutely continuous on a finite interval containing the origin.

(b) The function $\phi(x) = \sqrt{x}$, $x \in [0, c]$ *is absolutely continuous but not Lipschitz continuous.*

Proposition 3.1 ([Nie97]). *Absolutely continuous function is differentiable almost everywhere.*

Definition 3.2

An absolutely continuous function $t \mapsto \boldsymbol{\phi}(t, \boldsymbol{x}_0, \boldsymbol{u})$ is called a solution to the problem (3.1) for a given initial condition $\boldsymbol{x}_0 \in \mathbb{R}^n$ and a given input $\boldsymbol{u} \in \mathscr{U}$ if $\boldsymbol{\phi}(0, \boldsymbol{x}_0, \boldsymbol{u}(0)) = \boldsymbol{x}_0$ and $\dot{\boldsymbol{\phi}} = \boldsymbol{f}(\boldsymbol{\phi}, \boldsymbol{u})$ holds almost everywhere.

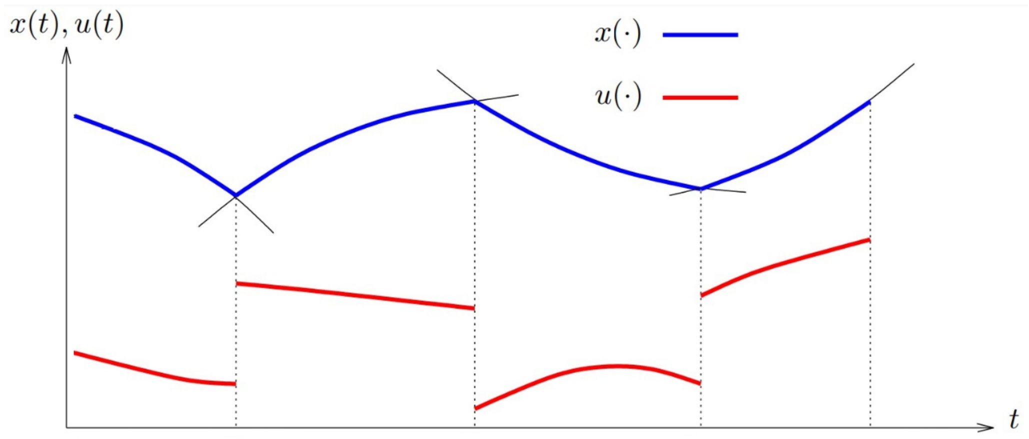

Figure 3.1: An illustration of an admissible input and the corresponding solution to (3.1).

Theorem 3.1

Let $\boldsymbol{f}$ be Lipschitz continuous with respect to the first argument uniformly with respect to the second argument, i.e., for any $w > 0$ there exists $L(w) > 0$ such that for any $\boldsymbol{x}, \boldsymbol{y} : |\boldsymbol{x}| \le w, |\boldsymbol{y}| \le w$ and for any $\boldsymbol{v} \in \mathbb{R}^m$ it holds that $|\boldsymbol{f}(\boldsymbol{y}, \boldsymbol{v}) - \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{v})| \le L(w)|\boldsymbol{y} - \boldsymbol{x}|$. Then, for any input $\boldsymbol{u} \in \mathcal{U}$ and any initial condition $\boldsymbol{x}_0 \in \mathbb{R}^n$ there exists a unique solution $\boldsymbol{x} = \boldsymbol{\phi}(\cdot, \boldsymbol{x}_0, \boldsymbol{u})$ to (3.1).

Comparison functions and their properties

In order to define the global stability properties of solutions to (3.1) we recall the standard classes of comparison functions:

$$
\begin{aligned}
\mathcal{P} &:= \{\gamma : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0} \,|\, \gamma \text{ is continuous}, \gamma(0) = 0 \text{ and } \gamma(r) > 0 \text{ for } r > 0\},\\
\mathcal{K} &:= \{\gamma \in \mathcal{P} \,|\, \gamma \text{ is strictly increasing}\},\\
\mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} \,|\, \gamma \text{ is unbounded}\},\\
\mathcal{L} &:= \{\gamma : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0} \,|\, \gamma \text{ is continuous and strickly decreasing with, } \lim_{t\to\infty} \gamma(t) = 0\},\\
\mathcal{KL} &:= \{\beta : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0} \,|\, \beta \text{ is continuous, } \beta(\cdot, t) \in \mathcal{K} \text{ for all } t \ge 0, \beta(r, \cdot) \in \mathcal{L} \text{ for all } r > 0\}.
\end{aligned}
$$

Example 3.2. *Examples of comparison functions:*

(a) *Linear function* $\gamma(r) = \alpha r$ *for some* $\alpha > 0$ *belongs to the classes* $\mathcal{P}, \mathcal{K}, \mathcal{K}_\infty$;

(b) *The function* $\gamma(r) = \frac{r}{r+1}$ *is of class* $\mathcal{K}$ *but not of class* $\mathcal{K}_\infty$ *since* $\lim_{r\to\infty} \frac{r}{r+1} = 1 < \infty$;

(c) *The function* $\beta(r, t) = M r e^{-\lambda t}$ *for some* $M, \lambda > 0$ *is of class* $\mathcal{KL}$.

Comparison functions can be used instead of "$\varepsilon - \delta$"-language to characterize the global stability properties of zero equilibrium to the ODE system (1.1) discussed in Chapter 1.

Definition 3.3

System (1.1) is called

- *globally stable* if there exists $\sigma \in \mathcal{K}_\infty$ such that for all $\boldsymbol{x}_0 \in \mathbb{R}^n$ the corresponding solution satisfies

$$|\boldsymbol{\phi}(t, \boldsymbol{x}_0)| \le \sigma(|\boldsymbol{x}_0|) \quad \text{for all} \quad t \ge 0; \tag{3.3}$$

- *globally asymptotically stable* (uniformly with respect to the states) if there exists $\beta \in \mathcal{KL}$ such that for all $\boldsymbol{x}_0 \in \mathbb{R}^n$ the corresponding solution satisfies

$$|\boldsymbol{\phi}(t, \boldsymbol{x}_0)| \le \beta(|\boldsymbol{x}_0|, t) \quad \text{for all} \quad t \ge 0. \tag{3.4}$$

Exercise 3.1. *(a) Let $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ be such that $\gamma_1 \circ \gamma_2(r) < r$ for all $r > 0$. Prove that $\gamma_2 \circ \gamma_1(r) < r$ for all $r > 0$.*

(b) Let $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ be such that $\gamma_1 \circ \gamma_2 \circ \gamma_3(r) < r$ for all $r > 0$. Is one of the following true?

1. $\gamma_3 \circ \gamma_1 \circ \gamma_2(r) < r$ *for all* $r > 0$;
2. $\gamma_2 \circ \gamma_3 \circ \gamma_1(r) < r$ *for all* $r > 0$;
3. $\gamma_1 \circ \gamma_3 \circ \gamma_2(r) < r$ *for all* $r > 0$.

ISS control systems

In this section, we discuss the stability concept that unifies the internal Lyapunov-type stability of control system and its robustness with respect to the external disturbances.

Definition 3.4: [Son89]

System (3.1) is called *input-to-state stable* (ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial value $\boldsymbol{x}_0 \in \mathbb{R}^n$ and any input $\boldsymbol{u} \in \mathcal{U}$ the corresponding solution $\boldsymbol{x} = \boldsymbol{\phi}(\cdot, \boldsymbol{x}_0, \boldsymbol{u})$ exists on $[0, \infty)$ and satisfies

$$|\boldsymbol{\phi}(t, \boldsymbol{x}_0, \boldsymbol{u})| \le \beta(|\boldsymbol{x}_0|, t) + \gamma(\|\boldsymbol{u}\|_\infty) \quad \text{for all} \quad t \ge 0. \tag{3.5}$$

Function $\gamma \in \mathcal{K}$ is called *ISS-gain* and describes the influence of the input $\boldsymbol{u} \in \mathcal{U}$ on the solutions of the system. Function $\beta \in \mathcal{KL}$ describes the transient behavior of the system.

Definition 3.5

System (3.1) is called *globally asymptotically stable at zero* (0-GAS) if the system (3.1) with $\boldsymbol{u} \equiv \boldsymbol{0}$ is GAS. System (3.1) possesses the *asymptotic gain property* (AG) if there exist $\gamma \in \mathcal{K}$ such that for any initial value $\boldsymbol{x}_0 \in \mathbb{R}^n$ and any input $\boldsymbol{u} \in \mathcal{U}$ it holds that

$$\limsup_{t \to \infty} |\boldsymbol{\phi}(t, \boldsymbol{x}_0, \boldsymbol{u})| \le \gamma(\|\boldsymbol{u}\|_\infty). \tag{3.6}$$

From (3.5), it immediately follows that the ISS implies 0-GAS and AG properties. Moreover, the converse statement is also true.

Theorem 3.2: [SW96]

System (3.1) is ISS if and only if it is 0-GAS and AG.

The following example shows that 0-GAS is not sufficient to conclude ISS.

Example 3.3 ([Son08]). *Consider a scalar nonlinear system*

$$\dot{x} = -x + (x^2 + 1)u. \tag{3.7}$$

System (3.7) *is 0-GAS, since it reduces to* $\dot{x} = -x$ *when* $u \equiv 0$. *On the other hand, solutions diverge even for some inputs that converge to zero. For example,*

$$\phi(t, \sqrt{2}, (2t+2)^{-\frac{1}{2}}) = \sqrt{2t+2} \xrightarrow{t\to\infty} \infty.$$

Hence, (3.7) *does not possess AG property. Even worse, the bounded input* $u \equiv 1$ *steers the state to infinity in a finite time.*

Exercise 3.2. *Prove that Definition 3.4 is equivalent to the existence of* $\tilde{\beta} \in \mathcal{KL}$ *and* $\tilde{\gamma} \in \mathcal{K}$ *such that for any initial value* $\boldsymbol{x}_0 \in \mathbb{R}^n$ *and any input* $\boldsymbol{u} \in \mathcal{U}$ *the corresponding solution* $\boldsymbol{x} = \boldsymbol{\phi}(\cdot, \boldsymbol{x}_0, \boldsymbol{u})$ *exists on* $[0, \infty)$ *and satisfies*

$$|\boldsymbol{\phi}(t, \boldsymbol{x}_0, \boldsymbol{u})| \le \max\left\{\tilde{\beta}(|\boldsymbol{x}_0|, t), \tilde{\gamma}(\|\boldsymbol{u}\|_\infty)\right\} \quad \textit{for all} \quad t \ge 0. \tag{3.8}$$

Hint: use the inequality $\max\{r, s\} \le r + s \le 2\max\{r, s\}$ *for all* $r, s \ge 0$.

Although the ISS is much stronger property than the 0-GAS for nonlinear systems (see Example 3.3), these properties are equivalent for linear control systems.

Theorem 3.3: [Son13]
A linear system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \tag{3.9}$$

is ISS if and only if it is 0-GAS.

Proof. (ISS ⇒ 0-GAS) Follows immediately from Definition 3.4 by substituting $\boldsymbol{u} \equiv \boldsymbol{0}$ into (3.5).

(0-GAS ⇒ ISS) The explicit solution to (3.9) with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and input $\boldsymbol{u} \in \mathcal{U}$ is given by

$$\boldsymbol{\phi}(t, \boldsymbol{x}, \boldsymbol{u}) = e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-s)} B\boldsymbol{u}(s)\,ds.$$

Since the (3.9) is 0-GAS, matrix A is Hurwitz and $\left\|e^{At}\right\| \le Me^{-\lambda t}$ for some $M, \lambda > 0$ and for all $t \ge 0$. Then,

$$\begin{aligned}
|\boldsymbol{\phi}(t, \boldsymbol{x}, \boldsymbol{u})| &\le |e^{At}\boldsymbol{x}_0| + \int_0^t \left\|e^{A(t-s)}\right\| |B\boldsymbol{u}(s)|\,ds \\
&\le Me^{-\lambda t}|\boldsymbol{x}_0| + M\int_0^t e^{-\lambda(t-s)}ds\,\|B\|\,\|\boldsymbol{u}\|_\infty \\
&\le Me^{-\lambda t}|\boldsymbol{x}_0| + M\frac{1}{\lambda}\left(1 - e^{-\lambda t}\right)\|B\|\,\|\boldsymbol{u}\|_\infty \\
&\le Me^{-\lambda t}|\boldsymbol{x}_0| + \frac{M\|B\|}{\lambda}\,\|\boldsymbol{u}\|_\infty.
\end{aligned}$$

Hence, system (3.9) satisfies Definition 3.4 with $\beta(r, t) = Mre^{-\lambda t}$ and $\gamma(r) = \frac{M\|B\|}{\lambda} r$. □

3.2 Lyapunov characterization of the ISS

This section presents a Lyapunov-based technique for the ISS verification.

Definition 3.6: [SW95b; JMW96]

A smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called an *ISS-Lyapunov function* for system (3.1) if

- there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|\boldsymbol{x}|) \leq V(\boldsymbol{x}) \leq \alpha_2(|\boldsymbol{x}|) \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n; \tag{3.10}$$

- there exist $\alpha \in \mathcal{P}$ and $\chi \in \mathcal{K}$ such that the following implication holds:

$$|\boldsymbol{x}| \geq \chi(|\boldsymbol{u}|) \quad \Rightarrow \quad \nabla V(\boldsymbol{x}) \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \leq -\alpha(|\boldsymbol{x}|) \tag{3.11}$$

  for all $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^m$.

If V is an ISS-Lyapunov function for (3.1), then V is a Lyapunov function (in the usual sense) for the autonomous system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{0})$.

Remark 3.1

Replacing the inequality (3.11) in Definition 3.6 with

$$V(\boldsymbol{x}) \geq \chi(|\boldsymbol{u}|) \quad \Rightarrow \quad \nabla V(\boldsymbol{x}) \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \leq -\alpha(V(\boldsymbol{x})) \tag{3.12}$$

leads to an equivalent definition of the ISS-Lyapunov function.

As an alternative to the implication form of the condition (3.11) in the Definition 3.6, one may use a dissipation type definition of the ISS-Lyapunov function:

Proposition 3.2 ([SW95b]). *A smooth function V is an ISS-Lyapunov function for* (3.1) *if and only if there exist $\alpha_i \in \mathcal{K}_\infty$ $(1 \leq i \leq 4)$ such that* (3.10) *holds, and*

$$\nabla V(\boldsymbol{x}) \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \leq -\alpha_3(|\boldsymbol{x}|) + \alpha_4(|\boldsymbol{u}|) \quad \textit{for all} \quad \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{u} \in \mathbb{R}^m.$$

Theorem 3.4: [SW95b; JMW96]

The system (3.1) is ISS if and only if it has an ISS-Lyapunov function.

Example 3.4. *Let us study the ISS property of the system*

$$\dot{x} = -x^3 + u, \tag{3.13}$$

where $x(t), u(t) \in \mathbb{R}$. As a candidate ISS-Lypaunov function we pick positive-definite proper function $V(x) = \frac{x^2}{2}$. Then,

$$\dot{V} = -x^4 + xu = -(1-\varepsilon)x^4 - \varepsilon x^4 + xu \leq -(1-\varepsilon)x^4 \quad \textit{if only} \quad |x| \geq \left(\frac{|u|}{\varepsilon}\right)^{\frac{1}{3}}$$

for any $\varepsilon \in (0,1)$. Then, V is an ISS-Lyapunov function for system (3.13) *in sense of Definition 3.6 with*

$$\alpha(r) = -(1-\varepsilon)r \quad \textit{and} \quad \chi(r) = \left(\frac{r}{\varepsilon}\right)^{\frac{1}{3}}.$$

From Theorem 3.4 we conclude the ISS of (3.13).

Exercise 3.3. *Let $x(t), u(t) \in \mathbb{R}$.*

(a) Check the ISS property of the system

$$\dot{x} \;=\; -x - 2x^3 + (1+x^2)u^2;$$

(b) Prove that the water tank system

$$\dot{x} \;=\; -\sqrt{2gx} + u$$

is ISS for some positive constant $g > 0$.

Exercise 3.4. *Let $x_1(t), x_2(t), u(t) \in \mathbb{R}$. Check the ISS property of the system*

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2^2,\\ \dot{x}_2 &= -x_2 + u.\end{aligned}$$

3.3 ISS feedback redesign

Consider a system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}), \quad t \geq 0 \tag{3.14}$$

with $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathcal{U}$, and $\boldsymbol{d} \in \mathcal{D} := L_\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^p)$ for some $n, p \in \mathbb{N}$. System (3.14) possesses two inputs:

- control input u, which we can choose (design) in order to archive some goal;
- external disturbance d, which we cannot influence.

Definition 3.7

System (3.14) is called *ISS stabilizable* if there exists a feedback $\boldsymbol{k} : \mathbb{R}^n \to \mathbb{R}^m$ such that system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{k}(\boldsymbol{x}), \boldsymbol{d})$$

is ISS with respect to the disturbance $\boldsymbol{d}$.

Definition 3.8

System (3.14) is called *GAS stabilizable* if there exists a feedback $\boldsymbol{k} : \mathbb{R}^n \to \mathbb{R}^m$ such that system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{k}(\boldsymbol{x}), \boldsymbol{0})$$

is GAS.

Example 3.5. *Consider a stabilization problem of the following system*

$$\dot{x} = x + (x^2+1)(u+d). \tag{3.15}$$

The term d may be interpreted as the actuator disturbance in the system. First, assume that $d \equiv 0$ and pick the feedback $k(x) = -\frac{2x}{1+x^2}$. Then, the closed-loop system takes the form

$$\dot{x} = -x. \tag{3.16}$$

System (3.16) *is GAS. Hence,* (3.15) *is GAS stabilizable. However, in the presence of actuator disturbances this controller leads to the system* $\dot{x} = -x + (x^2+1)d$*. As seen before in Example 3.3, this system has solutions which diverge to infinity even for disturbances* d *that converge to zero. Moreover, the constant disturbance* $d \equiv 1$ *results in solutions that explode in finite time. The chosen feedback* k *does not stabilize the system in sense of ISS. A natural question arise, whether it is possible to modify the feedback controller* k *so that the resulting system is ISS with respect to the disturbances. The answer is given in Theorem 3.5.*

Theorem 3.5: [Son89]

Consider a control-affine system

$$\dot{\boldsymbol{x}} = \boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})(u+d) \tag{3.17}$$

with $\boldsymbol{x}(t) \in \mathbb{R}^n$, $u(t), d(t) \in \mathbb{R}$ and suppose that there is some differentiable feedback law $u = k(\boldsymbol{x})$ so that

$$\dot{\boldsymbol{x}} = \boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})k(\boldsymbol{x})$$

has $\boldsymbol{x} = \boldsymbol{0}$ as a GAS equilibrium. Then, there is a feedback law $u = \tilde{k}(\boldsymbol{x})$ such that

$$\dot{\boldsymbol{x}} = \boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})(\tilde{k}(\boldsymbol{x}) + d)$$

is ISS with input d.

Proof. Since $\dot{\boldsymbol{x}} = \boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})k(\boldsymbol{x})$ is GAS there exists a smooth Lyapunov function V such that

$$\nabla V(x)\left(\boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})k(\boldsymbol{x})\right) \leq -\alpha(|\boldsymbol{x}|) \tag{3.18}$$

for some $\alpha \in \mathcal{K}_\infty$ and any $\boldsymbol{x} \in \mathbb{R}^n$. Let us estimate the Lie derivative of V with respect to the perturbed system (3.17) under the feedback controller $u = \tilde{k}(\boldsymbol{x})$:

$$\begin{aligned} \nabla V(x)\left(\boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})(\tilde{k}(\boldsymbol{x}) + d)\right) =& \nabla V(x)\left(\boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})k(\boldsymbol{x})\right) \\ &+ \nabla V(x)\boldsymbol{g}_1(\boldsymbol{x})\left(\tilde{k}(\boldsymbol{x}) - k(\boldsymbol{x}) + d\right). \end{aligned} \tag{3.19}$$

Choosing $\tilde{k}(\boldsymbol{x}) := k(\boldsymbol{x}) - \nabla V(x)\boldsymbol{g}_1(\boldsymbol{x})$, from (3.19), we get

$$\begin{aligned} \nabla V(x)\left(\boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})(\tilde{k}(\boldsymbol{x}) + d)\right) &\leq -\alpha(|\boldsymbol{x}|) - \left(\nabla V(\boldsymbol{x})\boldsymbol{g}_1(\boldsymbol{x})\right)^2 + \nabla V(\boldsymbol{x})\boldsymbol{g}_1(\boldsymbol{x})d \\ &\leq -\alpha(|\boldsymbol{x}|) - \left(\nabla V(\boldsymbol{x})\boldsymbol{g}_1(\boldsymbol{x})\right)^2 + \frac{1}{2}\left(\nabla V(\boldsymbol{x})\boldsymbol{g}_1(\boldsymbol{x})\right)^2 + \frac{1}{2}|d|^2 \\ &= -\alpha(|\boldsymbol{x}|) - \frac{1}{2}\left(\nabla V(\boldsymbol{x})\boldsymbol{g}_1(\boldsymbol{x})\right)^2 + \frac{1}{2}|d|^2 \end{aligned}$$

This means that V is an ISS-Lyapunov function of system $\dot{\boldsymbol{x}} = \boldsymbol{g}_0(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})(\tilde{k}(\boldsymbol{x}) + d)$ and the feedback law $u = \tilde{k}(\boldsymbol{x})$ is an ISS stabilizing controller for (3.17). □

Remark 3.2

GAS stabilizability with differentiable feedback feedback law implies ISS stabilizability also for a more general class of control-affine systems $\dot{\boldsymbol{x}} = \boldsymbol{g}_0(\boldsymbol{x}) + \sum_{i=1}^{m}(u_i + d_i)\boldsymbol{g}_i(x)$.

Example 3.6. *Find an ISS stabilizing feedback controller for the system*

$$\dot{x} = x + (x^2 + 1)u, \tag{3.20}$$

with $x(t), u(t) \in \mathbb{R}$ *in the presence of actuating disturbance* $d(t) \in \mathbb{R}$. *System* (3.20) *is GAS stabilizable. Indeed, let* $u = -\frac{2x}{x^2+1}$. *Then,* (3.20) *reduces to* $\dot{x} = -x$ *and it possesses a Lyapunov function* $V(x) = \frac{1}{2}x^2$. *According to Theorem 3.5 the following feedback controller*

$$u = -\frac{2x}{x^2+1} - x(x^2+1)$$

ISS stabilizes the system $\dot{x} = x + (x^2 + 1)(u + d)$.

ISS stabilization using backstepping

Consider a system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}_1(\boldsymbol{x})\boldsymbol{z} + \boldsymbol{G}_2(\boldsymbol{x})\boldsymbol{d}, \tag{3.21}$$

$$\dot{\boldsymbol{z}} = \boldsymbol{u} + \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{z})\boldsymbol{d}, \tag{3.22}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ and $\boldsymbol{z}(t) \in \mathbb{R}^m$ are the state vectors of system (3.21), (3.22), $\boldsymbol{u}(t) \in \mathbb{R}^m$ and $\boldsymbol{d} \in \mathbb{R}^p$ are the vectors of control and perturbation, all functions in the right-hand sides of (3.21), (3.22) are assumed to be locally Lipschitz continuous and $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$. It is assumed that there exists a smooth control law $\boldsymbol{k} : \mathbb{R}^n \to \mathbb{R}^m$ such that by plugging $\boldsymbol{z} = \boldsymbol{k}(\boldsymbol{x})$ into (3.21) the system becomes ISS in state $\boldsymbol{x}$ and input $\boldsymbol{d}$. Our problem is as follows: to design a new smooth control $\boldsymbol{u} = \tilde{\boldsymbol{k}}(\boldsymbol{x}, \boldsymbol{z})$ so that the entire system (3.21), (3.22) becomes ISS w.r.t. the disturbance $\boldsymbol{d}$. This requires to "transfer" the control law $\boldsymbol{z} = \boldsymbol{k}(\boldsymbol{x})$ through the integrator.

Theorem 3.6: [LSW99]

If system (3.21) is ISS stabilizable with a smooth control law $\boldsymbol{z} = \boldsymbol{k}(\boldsymbol{x})$ satisfying $\boldsymbol{k}(\boldsymbol{0}) = \boldsymbol{0}$, then the entire system (3.21), (3.22) is ISS stabilizable with a smooth control law $\boldsymbol{u} = \tilde{\boldsymbol{k}}(\boldsymbol{x}, \boldsymbol{z})$.

Proof. Since system (3.21) with the control law $\boldsymbol{z} = \boldsymbol{k}(\boldsymbol{x})$ is ISS, there exists a corresponding ISS-Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^p$

$$\alpha_1(|\boldsymbol{x}|) \leq V(\boldsymbol{x}) \leq \alpha_2(|\boldsymbol{x}|),$$

$$\nabla V(\boldsymbol{x}) \left(\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}_1(x)\boldsymbol{k}(\boldsymbol{x}) + \boldsymbol{G}_2(\boldsymbol{x})\boldsymbol{d} \right) \leq -\alpha(|\boldsymbol{x}|) + \sigma(|\boldsymbol{d}|)$$

for some functions $\alpha_1, \alpha_2, \alpha, \sigma \in \mathcal{K}_\infty$. Following [LSW99; DES11], we select

$$W(\boldsymbol{x}, \boldsymbol{z}) = V(\boldsymbol{x}) + \frac{1}{2}|\boldsymbol{z} - \boldsymbol{k}(\boldsymbol{x})|^2.$$

Its total time derivative is given by

$$\begin{aligned}
\dot{W}(\boldsymbol{x}, \boldsymbol{z}) &= \nabla V(\boldsymbol{x}) \left[\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}_1(\boldsymbol{x})\boldsymbol{z} + \boldsymbol{G}_2(\boldsymbol{x})\boldsymbol{d} \right] \\
&\quad + [\boldsymbol{z} - \boldsymbol{k}(\boldsymbol{x})] \left[\boldsymbol{u} + \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{z})\boldsymbol{d} - \nabla \boldsymbol{k}(\boldsymbol{x})[\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}_1(\boldsymbol{x})\boldsymbol{z} + \boldsymbol{G}_2(\boldsymbol{x})\boldsymbol{d}] \right] \\
&\leq -\alpha(|\boldsymbol{x}|) + \sigma(|\boldsymbol{d}|) \\
&\quad + [\boldsymbol{z} - \boldsymbol{k}(\boldsymbol{x})] \left[\boldsymbol{u} + \nabla V(\boldsymbol{x})\boldsymbol{G}_1(\boldsymbol{x}) + \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{z})\boldsymbol{d} - \nabla \boldsymbol{k}(\boldsymbol{x})[\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}_1(\boldsymbol{x})\boldsymbol{z} + \boldsymbol{G}_2(\boldsymbol{x})\boldsymbol{d}] \right].
\end{aligned}$$

Then, choosing feedback control

$$\boldsymbol{u} = \nabla k(\boldsymbol{x})[\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}_1(\boldsymbol{x})\boldsymbol{z}] - \nabla V(\boldsymbol{x})\boldsymbol{G}_1(\boldsymbol{x}) - [\boldsymbol{z} - \boldsymbol{k}(\boldsymbol{x})]\left[1 + |\boldsymbol{F}(\boldsymbol{x},\boldsymbol{z})|^2 + |\nabla \boldsymbol{k}(\boldsymbol{x})\boldsymbol{G}_2(\boldsymbol{x})|^2\right]$$

we conclude that

$$\dot{W}(\boldsymbol{x},\boldsymbol{z}) \le -\alpha(|\boldsymbol{x}|) - [\boldsymbol{z} - \boldsymbol{k}(\boldsymbol{x})]^2 + \sigma(|\boldsymbol{d}|) + |\boldsymbol{d}|^2,$$

which implies the ISS for (3.21), (3.22). □

3.4 Cascade and feedback interconnections

Consider a cascade

$$\begin{aligned} \dot{\boldsymbol{z}} &= \boldsymbol{f}(\boldsymbol{z},\boldsymbol{x}), \\ \dot{\boldsymbol{x}} &= \boldsymbol{g}(\boldsymbol{x},\boldsymbol{u}), \end{aligned} \tag{3.23}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{z}(t) \in \mathbb{R}^p$, $\boldsymbol{u} \in \mathcal{U}$ with $\boldsymbol{u}(t) \in \mathbb{R}^m$ for some $n, p, m \in \mathbb{N}$.

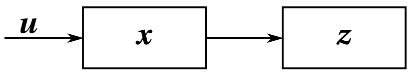

Figure 3.2: Cascade.

> **Theorem 3.7: [Son08]**
> Let each subsystem in (3.23) be ISS. Then, the cascade (3.23) is ISS.

Corollary 3.1. *In the special case in which the $\boldsymbol{x}$-subsystem has no inputs, the cascade of a GAS and an ISS system is GAS.*

Exercise 3.5. *Prove that the system from the Example 3.4 is ISS using cascade arguments.*

Next, we consider a more general type of feedback interconnection

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{f}_1(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{u}_1), \\ \dot{\boldsymbol{x}}_2 &= \boldsymbol{f}_2(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{u}_2), \end{aligned} \tag{3.24}$$

where, for $i = 1,2$, $\boldsymbol{x}_i(t) \in \mathbb{R}^{n_i}$, $\boldsymbol{u}_i(t) \in \mathbb{R}^{m_i}$ for all $t \ge 0$, and $\boldsymbol{f}_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$ is locally Lipschitz. We aim to relate stability properties of $\boldsymbol{x}_1$- and $\boldsymbol{x}_2$-subsystems with the stability of the whole interconnection (3.24).

Assume that, for $i = 1,2$, there exists an ISS-Lyapunov function V_i for the $\boldsymbol{x}_i$-subsystem such that the following holds:

- there exist functions $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ so that

$$\psi_{i1}(|\boldsymbol{x}_i|) \le V_i(\boldsymbol{x}_i) \le \psi_{i2}(|\boldsymbol{x}_i|) \tag{3.25}$$

  for all $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$;

- there exist functions $\alpha_i \in \mathcal{K}_\infty$, $\chi_i, \gamma_i \in \mathcal{K}$ so that the implications

$$V_1(\boldsymbol{x}_1) \ge \max\{\chi_1(V_2(\boldsymbol{x}_2)), \gamma_1(|\boldsymbol{u}_1|)\} \quad \Rightarrow \quad \nabla V_1(\boldsymbol{x}_1)\boldsymbol{f}_1(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{u}_1) \le -\alpha_1(V_1(\boldsymbol{x}_1)) \tag{3.26}$$

and

$$V_2(\boldsymbol{x}_2) \geq \max\{\chi_2(V_1(\boldsymbol{x}_1)), \gamma_2(|\boldsymbol{u}_2|)\} \quad \Rightarrow \quad \nabla V_2(\boldsymbol{x}_2)\boldsymbol{f}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}_2) \leq -\alpha_2(V_2(\boldsymbol{x}_2)) \tag{3.27}$$

hold for all $\boldsymbol{x}_i(t) \in \mathbb{R}^{n_i}$, $\boldsymbol{u}_i(t) \in \mathbb{R}^{m_i}$.

The following result provides a nonlinear small-gain condition under which an ISS-Lyapunov functions for the interconnected system (3.24) may be expressed in terms of ISS-Lyapunov functions for two subsystems.

Theorem 3.8: [JMW96]
Assume that, for $i = 1, 2$, the $\boldsymbol{x}_i$-subsystem has an ISS-Lyapunov function V_i satisying (3.25), (3.26) and (3.27). If

$$\chi_2 \circ \chi_1(r) < r \quad \text{for all} \quad r > 0 \tag{3.28}$$

then the interconnected system (3.24) is ISS. Moreover, there exist

- a $\mathcal{K}_\infty$-function σ continuously differentiable on $(0, \infty)$ with $\sigma'(r) > 0$ for all $r > 0$ such that

$$\chi_2(r) < \sigma(r) < \chi_1^{-1}(r) \quad \text{for all} \quad r > 0;$$

- a locally Lipschitz on $\mathbb{R}^{n_1+n_2} \setminus \{0\}$ function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ defined by

$$V(\boldsymbol{x}_1, \boldsymbol{x}_2) = \max\{\sigma(V_1(\boldsymbol{x}_1)), V_2(\boldsymbol{x}_2)\}$$

  such that for almost all $\boldsymbol{x}_1 \in \mathbb{R}^{n_1}$, $\boldsymbol{x}_2 \in \mathbb{R}^{n_2}$ and for all $\boldsymbol{u}_1 \in \mathbb{R}^{m_1}$, $\boldsymbol{u}_2 \in \mathbb{R}^{m_2}$

$$\nabla V(\boldsymbol{x}) \begin{pmatrix} \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}_1) \\ \boldsymbol{f}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}_2) \end{pmatrix} \leq -\alpha(V(\boldsymbol{x}_1, \boldsymbol{x}_2)) \quad \text{whenever} \quad V(\boldsymbol{x}_1, \boldsymbol{x}_2) \geq \gamma(|(\boldsymbol{u}_1, \boldsymbol{u}_2)^\top|)$$

  for some $\alpha \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}$.

Remark 3.3
The existence of locally Lipschitz function V (not necessarily smooth) is sufficient for the ISS [JMW96].

Exercise 3.6. *Let $x_1(t), x_2(t) \in \mathbb{R}$. Using the small-gain theorem find the values of the parameter $a \in \mathbb{R}$ so that the system*

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2, \\ \dot{x}_2 &= -x_2 + a\sqrt{|x_1|} \end{aligned} \tag{3.29}$$

is GAS.

Exercise 3.7. *Let $x_1(t), x_2(t), u_1(t), u_2(t) \in \mathbb{R}_{\geq 0}$. Find the values of the parameter $s \in \mathbb{R}_{\geq 0}$ so that the system*

$$\begin{aligned} \dot{x}_1 &= -\sqrt{x_1} + u_1 + s\sqrt{x_2}, \\ \dot{x}_2 &= -\sqrt{x_2} + u_2 + s\sqrt{x_1} \end{aligned} \tag{3.30}$$

is ISS.

3.5 ISS-related stability notions

In this section we provide an overview of some stability concepts that are weaker than the ISS.

Input-to-state stability with respect to a set

Let $\mathcal{A} \subset \mathbb{R}^n$ be a nonempty compact set. The distance from any point $\boldsymbol{x} \in \mathbb{R}^n$ to $\mathcal{A}$ is defined by $|\boldsymbol{x}|_{\mathcal{A}} := \inf_{\boldsymbol{y}\in\mathcal{A}} |\boldsymbol{x}-\boldsymbol{y}|$.

Definition 3.9: [SW95a]

System (3.1) is called *input-to-state stable with respect to* $\mathcal{A}$ if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial value $\boldsymbol{x}_0 \in \mathbb{R}^n$ and any input $\boldsymbol{u} \in \mathcal{U}$ the corresponding solution $\boldsymbol{x} = \boldsymbol{\phi}(\cdot, \boldsymbol{x}_0, \boldsymbol{u})$ exists on $[0,\infty)$ and satisfies

$$|\boldsymbol{\phi}(t,\boldsymbol{x}_0,\boldsymbol{u})|_{\mathcal{A}} \le \beta(|\boldsymbol{x}_0|_{\mathcal{A}}, t) + \gamma(\|\boldsymbol{u}\|_\infty) \quad \text{for all} \quad t \ge 0. \tag{3.31}$$

The ISS w.r.t $\mathcal{A}$ property for the system (3.1) is equivalent to the existence of a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ satisfying

$$\begin{aligned}
&\alpha_1(|\boldsymbol{x}|_{\mathcal{A}}) \le V(\boldsymbol{x}) \le \alpha_2(|\boldsymbol{x}|_{\mathcal{A}}) \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n \\
&|\boldsymbol{x}|_{\mathcal{A}} \ge \alpha_3(|\boldsymbol{u}|) \quad \Rightarrow \quad \nabla V(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) \le -\alpha_4(V(\boldsymbol{x})) \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{u} \in \mathbb{R}^m
\end{aligned}$$

for some $\alpha_i \in \mathcal{K}_\infty$ $(i = 1, \dots, 4)$.

Exercise 3.8. *Prove that the system*

$$\dot{x} = x^2 - x^3 + u$$

with $x(t), u(t) \in \mathbb{R}$ *is ISS w.r.t.* $\mathcal{A} = [0, 1]$.

Integral input-to-state stability

Definition 3.10: [Son98]

System (3.1) is called *integral input-to-state stable* (iISS) if there exist $\beta \in \mathcal{KL}$ and $\alpha, \gamma \in \mathcal{K}_\infty$ such that for any initial value $\boldsymbol{x}_0 \in \mathbb{R}^n$ and any input $\boldsymbol{u} \in \mathcal{U}$ the corresponding solution $\boldsymbol{x} = \boldsymbol{\phi}(\cdot, \boldsymbol{x}_0, \boldsymbol{u})$ exists on $[0,\infty)$ and satisfies

$$\alpha(|\boldsymbol{\phi}(t,\boldsymbol{x}_0,\boldsymbol{u})|) \le \beta(|\boldsymbol{x}_0|, t) + \int_0^t \gamma(|\boldsymbol{u}(s)|)\,ds \quad \text{for all} \quad t \ge 0. \tag{3.32}$$

Exercise 3.9. *Show that the system*

$$\dot{x} = -x + xu$$

with $x(t), u(t) \in \mathbb{R}$ *is iISS, but not ISS.*

The iISS property for the system (3.1) is equivalent to the existence of a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ satisfying

$$\begin{aligned}
&\alpha_1(|\boldsymbol{x}|) \le V(\boldsymbol{x}) \le \alpha_2(|\boldsymbol{x}|) \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n \\
&\nabla V(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) \le -\alpha_3(|\boldsymbol{x}|) + \alpha_4(|\boldsymbol{u}|) \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{u} \in \mathbb{R}^m
\end{aligned}$$

for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 \in \mathcal{P}$, and $\alpha_4 \in \mathcal{K}_\infty$.

In contrast to the ISS case (see Proposition 3.2), where α_3 is required to be of class $\mathcal{K}_\infty$, it is sufficient to use positive definite α_3 for the iISS characterization. Also, there is no characterization for the iISS property via Lyapunov-like function in implication form similar to the ISS case (see Theorem 3.4).

Local input-to-state stability

Definition 3.11

System (3.1) is called *locally input-to-state stable* (LISS) if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$, and constants $\rho_x, \rho_u > 0$ such that for any initial value $|\boldsymbol{x}_0| \leq \rho_x$ and any input $\boldsymbol{u} \in \mathcal{U}$ with $\|\boldsymbol{u}\|_\infty \leq \rho_u$ the corresponding solution $\boldsymbol{x} = \boldsymbol{\phi}(\cdot, \boldsymbol{x}_0, \boldsymbol{u})$ exists on $[0, \infty)$ and satisfies

$$|\boldsymbol{\phi}(t, \boldsymbol{x}_0, \boldsymbol{u})| \leq \beta(|\boldsymbol{x}_0|, t) + \gamma(\|\boldsymbol{u}\|_\infty) \quad \text{for all} \quad t \geq 0. \tag{3.33}$$

The LISS property for system (3.1) is equivalent to the existence of a smooth function $V : D \to \mathbb{R}_{\geq 0}$, $0 \in \mathtt{int}(D) \subset \mathbb{R}^n$ satisfying conditions (3.10), (3.11) for all $\boldsymbol{x} \in D$, $u \in U := \{\boldsymbol{u} \in \mathbb{R}^m : |u| \leq \rho\}$ for some constant $\rho > 0$.

From Definitions 3.4, 3.5, 3.10, and 3.11, it follows that

$$\text{ISS} \subset \text{iISS} \subset 0-\text{GAS} \quad \text{and} \quad \text{ISS} \subset \text{iISS} \subset \text{LISS}.$$

Exercise 3.10. *Show by means of counterexamples that*

$$\textit{LISS} \not\subset 0 - \textit{GAS}.$$

References

[DES11] SN Dashkovskiy, Denis V Efimov, and Eduardo D Sontag. „Input to state stability and allied system properties“. In: *Automation and Remote Control* 72.8 (2011), p. 1579 (cit. on p. 28).

[JMW96] Zhong-Ping Jiang, Iven MY Mareels, and Yuan Wang. „A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems“. In: *Automatica* 32.8 (1996), pp. 1211–1215 (cit. on pp. 25, 30).

[LSW99] Daniel Liberzon, Eduardo D Sontag, and Yuan Wang. „On integral-input-to-state stabilization“. In: *Proceedings of the 1999 American Control Conference (Cat. No. 99CH36251)*. Vol. 3. IEEE. 1999, pp. 1598–1602 (cit. on p. 28).

[Nie97] Ole Nielson. *An introduction to integration and measure theory*. John Wiley & Sons, Inc., New York., 1997 (cit. on p. 21).

[Son08] Eduardo D Sontag. „Input to state stability: Basic concepts and results“. In: *Nonlinear and optimal control theory*. Springer, 2008, pp. 163–220 (cit. on pp. 23, 29).

[Son13] Eduardo D Sontag. *Mathematical control theory: deterministic finite dimensional systems*. Vol. 6. Springer Science & Business Media, 2013 (cit. on p. 24).

[Son89] Eduardo D Sontag. „Smooth stabilization implies coprime factorization“. In: *IEEE Transactions on Automatic Control* 34.4 (1989), pp. 435–443 (cit. on pp. 23, 27).

[Son98] Eduardo D Sontag. „Comments on integral variants of ISS“. In: *Systems & Control Letters* 34.1-2 (1998), pp. 93–100 (cit. on p. 31).

[SW95a] Eduardo D. Sontag and Yuan Wang. „On Characterizations of Input-to-State Stability with Respect to Compact Sets". In: *IFAC Proceedings Volumes* 28.14 (1995). 3rd IFAC Symposium on Nonlinear Control Systems Design 1995, Tahoe City, CA, USA, 25-28 June 1995, pp. 203–208 (cit. on p. 31).

[SW95b] Eduardo D Sontag and Yuan Wang. „On characterizations of the input-to-state stability property". In: *Systems & Control Letters* 24.5 (1995), pp. 351–359 (cit. on p. 25).

[SW96] Eduardo D Sontag and Yuan Wang. „New characterizations of input-to-state stability". In: *IEEE Transactions on Automatic Control* 41.9 (1996), pp. 1283–1294 (cit. on p. 23).