Advanced Methods in Nonlinear Control

Lecture Notes

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Lyapunov stability and Lyapunov's direct method 1

1.1 Stability in the sense of Lyapunov

Consider the autonomous nonlinear dynamical system

$$\dot{\boldsymbol{x}}x = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{1.1}$$

with solutions

$$\boldsymbol{x}(t) = \boldsymbol{x}(t; \boldsymbol{x}_0).$$

A fundamental property of linear and nonlinear systems is the stability of *equilibrium points*, i.e. solutions $\boldsymbol{x}^*$ of the algebraic equation

$$\boldsymbol{0} = \boldsymbol{f}(\boldsymbol{x}^*, \boldsymbol{x}p). \tag{1.2}$$

Throughout this text we will be concerned with stability and thus we need a formal definition. Here, we employ the common definitions associated to A. Lyapunov [Ful92; Kha96].

Definition 1.1

An equilibrium point $\boldsymbol{x}^*$ of (1.1) is said to be *stable*, if for any $\epsilon > 0$ there exists a constant $\delta > 0$ such that for any initial deviation from equilibrium within a δ-neighborhood, the trajectory is comprised within an ϵ-neighborhood, i.e.

$$\forall\, \boldsymbol{x}_0 : ||\boldsymbol{x}_0 - \boldsymbol{x}^*|| \leq \delta \Rightarrow ||\boldsymbol{x}(t) - \boldsymbol{x}^*|| \leq \epsilon\, \forall\, t \geq 0. \tag{1.3}$$

If $\boldsymbol{x}^*$ is not stable, it is called *unstable*.

This concept is illustrated in Figure 1.1 (left), and is also referred to as *stability in the sense of Lyapunov*. Stability implies that the solutions stay arbitrarily close to the equilibrium whenever the initial condition is chosen sufficiently close to the equilibrium point. Note that stability implies boundedness of solutions, but not that these converge to an equilibrium point. Convergence in turn is ensured by the concept of attractivity.

Definition 1.2

The equilibrium point $\boldsymbol{x}^*$ is called *attractive* (or an *attractor*) for the set $\mathscr{D} \subseteq \mathbb{R}^n$, if

$$\forall\, \boldsymbol{x}_0 \in \mathscr{D} : \lim_{t\to\infty} ||\boldsymbol{x}(t) - \boldsymbol{x}^*|| = 0. \tag{1.4}$$

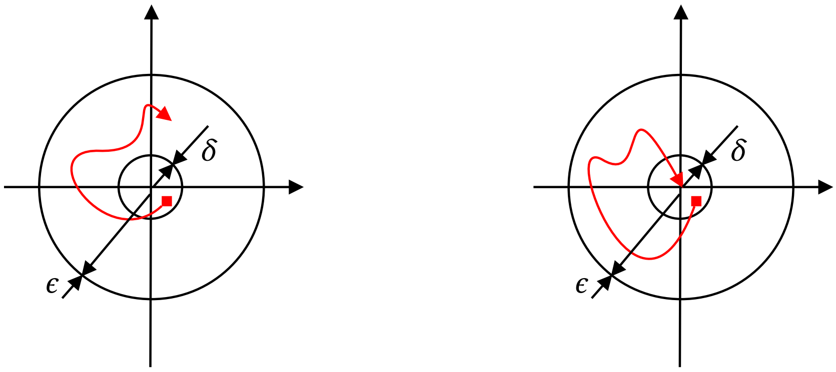

Figure 1.1: Qualitative illustration of the concepts of stability (left) and asymptotic stabilty (right).

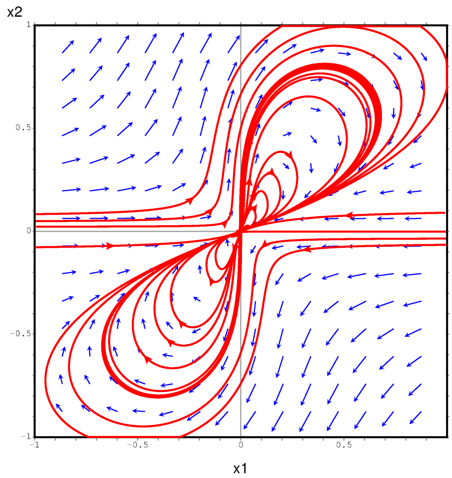

Figure 1.2: Phase portrait of the Vinograd system (1.5), with an unstable but attractive equilibrium point at $\boldsymbol{x}^* = \mathbf{0}$.

Note that an equilibrium point which instead of attracting the trajectories does repell them is called a *repulsor*. Obviously, a repulsor is unstable, given that for small ϵ and for any delta the trajectories starting in the δ neighborhood will eventually leave the ϵ neigborhood.

The set $\mathscr{D}$ is called the *domain of attraction*. Note that an equilibrium point may be attractive without being stable, i.e. that trajectories always have a large transient, so that for small ϵ no trajectory will stay for all times within the ϵ-neighborhood, but will return to it and converge to the equilibrium point. An example of such a behavior is given by Vinograd's system [Vin57]

$$\begin{aligned} \dot{x}_1 &= \frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)\left(1 + (x_1^2 + x_2^2)^2\right)} \\ \dot{x}_2 &= \frac{x_2^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)\left(1 + (x_1^2 + x_2^2)^2\right)} \end{aligned} \tag{1.5}$$

with phase portrait shown in Figure 1.2, illustrating a butterfly-shaped behavior where even for very small initial deviations from equilibrium there is a large transient returning asymptotically to the equilibrium point $\boldsymbol{x}^* = \mathbf{0}$. It should be mentioned, that if one can demonstrate convergence within a domain $\mathscr{D}$, this does not necessarily imply that $\mathscr{D}$ is the maximal domain of attraction. The determination of the maximal domain of attraction for an attractive equilibrium of a nonlinear system is, in general, a non-trivial task, due to the fact that, in contrast to linear systems, nonlinear systems can have multiple attractors, each one with its own domain of attraction. An example is given by the system

$$\dot{x} = x(1 - x^2), \quad x(0) = x_0$$

which has three equilibrium points, namely $x_1^* = -1, x_2^* = 0$ and $x_3^* = 1$, with $x_{1,3}^*$ being attractors and x_2^* being a repulsor. The domain of attraction of x_1^* is $\mathscr{D}_1 = (-\infty, 0)$ and of x_3^* is $\mathscr{D}_3 = (0, \infty)$.

The concept which combines the above two concepts is given by the asymptotic stability defined next.

Definition 1.3

An equilibrium point $\boldsymbol{x}^*$ of (1.1) is said to be *asymptotically stable* if it is stable and attractive.

This concept is illustrated in Figure 1.1 (right). Note that in contrast to pure attractivity, the concept of asymptotic stability does not allow for large transients associated to small initial deviations, and is thus much stronger, and more of practical interest.

It must be noted that the asymptotic stability does not state anything about convergence speed. It only establishes that after an infinite time period the solution $\boldsymbol{x}(t)$, if starting in the domain of attraction, will approach $\boldsymbol{x}^*$ without ever reaching it actually. For most practical applications it is desirable to be able to state how long it will take to reach the equilibrium point at least sufficiently close. In this context, sufficiently close is often associated to a 95% to 99% convergence. A concept which allows to establish explicit statement about convergence speed is the exponential stability.

Definition 1.4

An equilibrium point $\boldsymbol{x}^*$ of (1.1) is said to be *exponentially stable* in a set $\mathscr{D}$, if it is stable, and there are constants $a, \lambda > 0$ so that

$$\forall\, \boldsymbol{x}_0 \in \mathscr{D} : ||\boldsymbol{x}(t) - \boldsymbol{x}^*|| \leq a||\boldsymbol{x}_0 - \boldsymbol{x}^*||e^{-\lambda t}. \tag{1.6}$$

The constant a is known as the amplitude, and λ as the convergence rate.

Clearly, exponential stability implies asymptotic stability. Actually, the attractivity is ensured by exponential term and the stability follows from the fact that by transitivity of the inequality relation it holds that for all $\boldsymbol{x}_0$ so that

$$\|\boldsymbol{x}_0 - \boldsymbol{x}^*\| \leq \delta \leq \frac{\epsilon}{a}$$

it follows that

$$||\boldsymbol{x}(t) - \boldsymbol{x}^*|| \leq a||\boldsymbol{x}_0 - \boldsymbol{x}^*||e^{-\lambda t} \leq a||\boldsymbol{x}_0 - \boldsymbol{x}^*|| \leq a\delta \leq \epsilon$$

meaning that the trajectories stay for all $t \geq 0$ in the ϵ neighborhood.

The ratio

$$t_c = \frac{1}{\lambda}$$

is called the characteristic time constant for the exponential convergence assessment and it is straightforward to show that the bounding curve $\|\boldsymbol{x}_0 - \boldsymbol{x}\| e^{-\lambda t}$ converges to zero up to 98.5% within 4 characteristic times t_c.

Note that all the above stability and attractivity concepts can also be applied to sets. To examplify this, and for later reference, consider the definition of an attractive compact set.

Definition 1.5

A compact set $\mathcal{M}$ is called attractive for the domain $\mathcal{D}$, if

$$\forall\, \boldsymbol{x}_0 \in \mathcal{D}: \lim_{t\to\infty} \boldsymbol{x}(t) \in \mathcal{M}.$$

Convergence to a set is of practical interest because in many situations it may be used to obtain a reduced model for analysis and design purposes, or may even be part of the design as in the *sliding-mode control* approach (see e.g. [Utk92]). In many situations it is an essential part of the stability assessment of a dynamical system [Sei69]. Furthermore, in many applications it is more important to show the convergence to a set rather than to an equilibrium point.

Finally, it will be helpfull in the sequel to understand the concept of (positively) invariant sets $\mathcal{M}$ for a dynamical system.

Definition 1.6

A set $\mathcal{M} \subset \mathbb{R}^n$ is called *positively invariant*, if for all $\boldsymbol{x}_0 \in \mathcal{M}$ it holds that $\boldsymbol{x}(t;\boldsymbol{x}_0) \in \mathcal{M}$ for all $t \geq 0$.

The distinction positive invariant is used to distinguish the concept from negative invariance, referring to a reversion of time (i.e., letting time tending to minus infinity). The simplest example of a positively invariant set is an equilibrium point.

1.2 Lyapunov's direct method

A very usefull way to establish the stability of an equilibrium point for a nonlinear dynamical systems consists in Lyapunov's direct method. Motivated by studies on energy dissipation in physical processes, in particular in astronomy, Aleksandr Mikhailovich Lyapunov, generalized these considerations to functions which are positive for any non-zero argument [Ful92]. In the sequel consider that the equilibrium point under consideration is the origin $\boldsymbol{x} = \boldsymbol{0}$. If other equilibria have to analyzed a linear coordinate shift $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}^*$ can be employed to move the equilibrim to the origin in the coordinate $\tilde{\boldsymbol{x}}$.

To summarize the results of Lyapunov and generalizations of it some definitions are in order.

Definition 1.7

A continuous functional $V : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ is called

- positive semi-definite if $\forall\, \boldsymbol{x} : V(\boldsymbol{x}) \geq 0$.
- positive definite if $\forall\, \boldsymbol{x} \neq \boldsymbol{0} : V(\boldsymbol{x}) > 0$ and $V(\boldsymbol{x}) = 0$ only for $\boldsymbol{x} = \boldsymbol{0}$.
- negative semi-definite if $\forall\, \boldsymbol{x} : V(\boldsymbol{x}) \leq 0$.

- negative definite if $\forall\, \boldsymbol{x} \neq \mathbf{0} : V(\boldsymbol{x}) < 0$ and $V(\boldsymbol{x}) = 0$ only for $\boldsymbol{x} = \mathbf{0}$.

With these notions at hand, the following result can be stated.

Theorem 1.1

Let $V : \mathscr{D} \subset \mathbb{R}^n, V(\boldsymbol{x}) > 0$ be positive definite. If $\forall\, \boldsymbol{x} \in \mathscr{D} : \frac{dV}{dt}(\boldsymbol{x}) = \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}} \leq 0$, then $\boldsymbol{x} = \mathbf{0}$ is stable in the sense of Lyapunov.

Proof. Given that $V(\boldsymbol{x}) > 0$ and its continuity, there exists a function $W(\boldsymbol{x}) > 0$ such that

$$W(\boldsymbol{x}) \leq V(\boldsymbol{x}), \quad \forall\, \boldsymbol{x} \in \mathscr{D}. \tag{1.7}$$

Let $\epsilon > 0$ and set

$$m := \min_{||\boldsymbol{x}||=\epsilon} W(\boldsymbol{x}) > 0. \tag{1.8}$$

Choose $\delta > 0$ such that

$$\max_{0 \leq ||\boldsymbol{x}|| \leq \delta} V(\boldsymbol{x}) \leq m.$$

Given that $m > 0$, $V(\boldsymbol{x}) > 0$ and the continuity of V such a positive δ always exists. It follows from the fact that V is non-increasing over time ($\dot{V}(\boldsymbol{x}) < 0$) that

$$\forall\, \boldsymbol{x}_0 : ||\boldsymbol{x}_0|| \leq \delta \quad \Rightarrow \quad V(\boldsymbol{x}(t; \boldsymbol{x}_0)) \leq m.$$

By (1.7) this implies that

$$W(\boldsymbol{x}(t; \boldsymbol{x}_0)) \leq m.$$

By virtue of the definition of m in equation (1.8) it follows that

$$\forall\, \boldsymbol{x}_0 : ||\boldsymbol{x}_0|| \leq \delta \Rightarrow ||\boldsymbol{x}(t; \boldsymbol{x}_0)|| \leq \epsilon$$

showing that the origin is stable in the sense of Lyapunov. □

A function $V > 0$ that satisfies the conditions of Theorem 1.1 is called a *Lyapunov function*. Note that if $V > 0$ is continuously differentiable but it is not clear if $\frac{dV}{dt} \leq 0$ or the sign depends on some system parameters, then it is called a *Lyapunov function candidate*.

As can be seen from the proof, an essential part consists in that the sets

$$\Gamma_c = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, V(\boldsymbol{x}) = c\} \tag{1.9}$$

defined by level curves of $V(\boldsymbol{x})$ are the boundaries of compact subsets $\mathscr{D}_c$ of the state space. In virtue of the non-increasing nature of V these sets are positively invariant. The geomtric idea of the proof of Theorem 1.1 is quite beautiful and will be shortly discussed. See Figure 1.3 for an illustration. The conditions of the theorem ensure that for a given ϵ there exists a value $c > 0$ such that the set $\mathscr{D}_c$ with the boundary Γ_c defined in (1.9) is completely contained in the ϵ-neigborhood $\mathscr{N}_\epsilon$ of the origin, i.e. it holds that

$$\mathscr{D}_c \subseteq \mathscr{N}_\epsilon.$$

Choosing $\delta > 0$ such that the δ-neighborhood $\mathcal{N}_\delta$ is completely contained in $\mathcal{D}_c$ one obtains that

$$\mathcal{N}_\delta \subseteq \mathcal{D}_c \subseteq \mathcal{N}_\epsilon$$

with $\mathcal{D}_c$ being positively invariant. Thus it holds that for all $\boldsymbol{x}_0$ with $\|\boldsymbol{x}_0\| \leq \delta$, i.e. $\boldsymbol{x}_0 \in \mathcal{N}_\delta$ the solution $\boldsymbol{x}(t;\boldsymbol{x}_0)$ is contained in $\mathcal{D}_c \subset \mathcal{N}_\epsilon$, implying that $\|\boldsymbol{x}(t)\| \leq \epsilon$ for all $t \geq 0$.

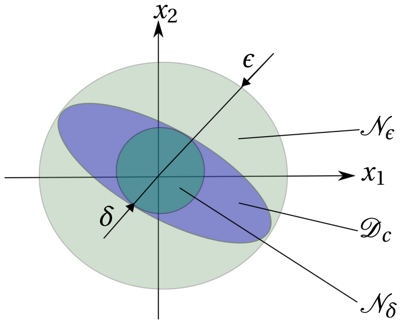

Figure 1.3: Geometrical idea behind the proof of Lyapunov's direct method in a two-dimensional state space.

The above only holds locally, unless $V(\boldsymbol{x})$ is strictly growing with $\|\boldsymbol{x}\|$. Thus the result is only local. The maximum compact set implied by the particular Lyapunov function can be explicitely determined. In the case that $\lim_{\|\boldsymbol{x}\|\to\infty} V(\boldsymbol{x}) = \infty$ the function is called *radially unbounded*. For a radially unbounded Lyapunov function the above result becomes global, i.e. it holds with $\mathcal{D} = \mathbb{R}^n$.

By evaluating explicitely the inequality $\frac{dV}{dt}(\boldsymbol{x}) = 0$ which holds over the set

$$\mathcal{X}_0 = \left\{\boldsymbol{x} \in \mathbb{R}^n \mid \frac{dV(\boldsymbol{x})}{dt} = 0\right\} \tag{1.10}$$

one can apply the following result going to back to Nikolay Nikolayevich Krasovsky and Joseph Pierre LaSalle and is known as the *invariance theorem*.

Theorem 1.2

(Krasovsky-LaSalle) Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a postively invariant compact set and $V \in \mathcal{C}^1(\mathcal{D} \to \mathbb{R})$ positive defininte function with $\frac{dV}{dt}(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x} \in \mathcal{D}$. Then the trajectories $\boldsymbol{x}(t)$ converge to the largest positively invariant set $\mathcal{M} \subseteq \mathcal{X}_0$ with $\mathcal{X}_0$ defined in (1.10).

If the conditions of this theorem are satisfied, an additional condition implies the asymptotic stability of the origin as stated next.

Theorem 1.3

If the conditions of Theorem 1.2 are satisfied and it holds that $\mathcal{M} = \{\boldsymbol{0}\}$, then the origin $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable.

A typical system where these results can be illustrated is given by the following Lienard oscillator

$$\ddot{x} + d\dot{x} + f(x) = 0 \tag{1.11}$$

with $d > 0$ and $f(x) > 0$ for $x > 0$, $f(x) = 0$ for $x = 0$ and $f(-x) = -f(x)$. The oscillator (1.11) can be written equivalently in state-space form with $x_1 = x$ and $x_2 = \dot{x}$ as

$$\dot{x}_1 = x_2 \tag{1.12a}$$
$$\dot{x}_2 = -f(x_1) - dx_2. \tag{1.12b}$$

Consider the following Lyapunov function candidate

$$V(\boldsymbol{x}) = \int_0^{x_1} f(\xi)d\xi + \frac{1}{2}x_2^2$$

motivated by the energy contained in the motion of $\boldsymbol{x}$ in form of potential and kinetic energy. The change in time of V is governed by

$$\begin{aligned}\frac{dV}{dt}(\boldsymbol{x}) &= f(x_1)\dot{x}_1 + x_2\dot{x}_2 \\ &= f(x_1)x_2 + x_2(-f(x_1) - dx_2) \\ &= -dx_2^2 \leq 0\end{aligned}$$

implying stability of the origin $\boldsymbol{x} = \boldsymbol{0}$ in virtue of Theorem 1.1. From Theorem 1.2 it is additionally known that $\boldsymbol{x}$ converges into the set

$$\mathscr{X}_0 = \{\boldsymbol{x} \in \mathbb{R}^2 \,|\, x_2 = 0\},$$

and more specifically into the largest positively invariant subset of $\mathscr{M} \subseteq \mathscr{X}_0$. This set in turn contains only trajectories for which $x_2(t) = 0$ for all times, given that it is positively invariant. This means that $\dot{x}_2(t) = 0$ for all times. Substituting $x_2 = 0, \dot{x}_2 = 0$ into (1.12b) this means that $f(x_1(t)) = 0$ for all times, showing that

$$\mathscr{M} = \{\boldsymbol{0}\}$$

given that $f(x_1) = 0$ only for $x_1 = 0$. Corollary 1.3 implies that the origin $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable.

The asymptotic stability can also be concluded using Lyapunov's direct method if $\frac{dV}{dt}(\boldsymbol{x})$ is negative definite. This is stated in the next theorem.

Theorem 1.4

Let $V : \mathscr{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n, V > 0$. If $\forall\, \boldsymbol{x} \in \mathscr{D} : \frac{dV}{dt}(\boldsymbol{x}) < 0$, then $\boldsymbol{x} = \boldsymbol{0}$ is locally asymptotically stable in $\mathscr{D}$.

Proof. In virtue of Theorem 1.1 we have that $\boldsymbol{x} = \boldsymbol{0}$ is stable in the sense of Lyapunov. It thus remains to show that $\lim_{t\to\infty} V(\boldsymbol{x}) = 0$ to conclude, by taking into account the positive definiteness and the continuity of $V(\boldsymbol{x})$, that $\lim_{t\to\infty} ||\boldsymbol{x}(t)|| = 0$.

Assume that V does not converge to zero. Then there exists a positive constant $c > 0$ such that $\lim_{t\to\infty} V(\boldsymbol{x}(t)) = c > 0$. Let

$$S = \{\boldsymbol{x} \in \mathscr{D} \,|\, V(\boldsymbol{x}) \leq c\}$$

By assumption, for $\boldsymbol{x}_0 \notin S$, i.e. $V(\boldsymbol{x}_0) > c$ it holds that $\forall\, t \geq 0 : \boldsymbol{x}(t) \notin S$. Let Γ_S be the boundary of the set S, i.e.

$$\Gamma_S = \{\boldsymbol{x} \in \mathscr{D} \,|\, V(\boldsymbol{x}) = c\}.$$

We have that $\dot{V}(\boldsymbol{x})|_{\boldsymbol{x}\in\Gamma_S} < 0$. Introduce

$$-\gamma := \max_{\boldsymbol{x}\in\Gamma_S} \dot{V}(\boldsymbol{x}) < 0.$$

Now, let $\boldsymbol{x}_0 \notin S$, i.e. $V(\boldsymbol{x}_0) > c$ and let $t > t_* := \frac{V(\boldsymbol{x}_0)-c}{\gamma}$. Observe that

$$\begin{aligned} V(\boldsymbol{x}(t;\boldsymbol{x}_0)) &= V(\boldsymbol{x}_0) + \int_0^t \dot{V}(\boldsymbol{x}(\tau;\boldsymbol{x}_0))d\tau \\ &\leq V(\boldsymbol{x}_0) - \gamma t < V(\boldsymbol{x}_0) - \gamma t_* = c \end{aligned}$$

implying that $\forall\, t > t_*$ it holds that $\boldsymbol{x}(t;\boldsymbol{x}_0) \in \mathscr{S}$. This contradicts the initial assumption that $\forall\, t \geq 0 : \boldsymbol{x}(t) \notin S$, and thus c cannot be positive and it must hold that $c = 0$. This, in turn, implies that $\lim_{t\to\infty} V(\boldsymbol{x}(t)) = 0$, and thus $\forall\, \boldsymbol{x}_0 \in \mathscr{D} : \lim_{t\to\infty} ||\boldsymbol{x}(t)|| = 0$. □

At this place it is noteworthy that using Lyapunov functions one can establish a domain for which the equilibrium point is an attractor. This domain will always be included in the domain of attraction of the equilibrium point. Note that eventhough it is not possible to conclude if the domain of attraction established in this way is the complete domain of attraction or only a subset of it, unless the result is global.

As discussed above, in may cases it is not sufficient to conclude only the asymptotic stability and it becomes important to have a quantitative value for the convergence speed towards an equilibrium. This can be established if the equilibrium is exponentially stable (see Definition 1.4). Exponential stability can be concluded using Lyapunov functions if some additional properties are given. These are stated in the next theorem.

Theorem 1.5

Let $V : \mathscr{D} \subset \mathbb{R}^n \to \mathbb{R}, V > 0$ be a positive definite functional. If there exist constants $\alpha, \beta, \gamma > 0$ so that

$$(i) \quad \alpha\|\boldsymbol{x}\|^2 \leq V(\boldsymbol{x}) \leq \beta\|\boldsymbol{x}\|^2 \tag{1.13a}$$

$$(ii) \quad \frac{dV}{dt}(\boldsymbol{x}) \leq -\gamma V(\boldsymbol{x}) \tag{1.13b}$$

then $\boldsymbol{x} = \boldsymbol{0}$ is exponentially stable and (1.6) holds with $a = \sqrt{\beta/\alpha}$ and $\lambda = \gamma/2$.

Proof. In virtue of (1.13b) it holds that

$$V(\boldsymbol{x}(t)) \leq V(\boldsymbol{x}_0)e^{-\gamma t}.$$

From (1.13a) this implies that

$$\|\boldsymbol{x}(t)\|^2 \leq \frac{1}{\alpha} V(\boldsymbol{x}(t)) \leq \frac{1}{\alpha} V(\boldsymbol{x}_0)e^{-\gamma t} \leq \frac{\beta}{\alpha} e^{-\gamma t}\|\boldsymbol{x}_0\|^2$$

and finally

$$\|\boldsymbol{x}(t)\| \leq \sqrt{\frac{\beta}{\alpha}}\|\boldsymbol{x}_0\| e^{-\frac{\gamma}{2}t}.$$

Exponentially stability as defined in (1.6) follows with a and λ stated above.

□

The fact that the value of $V(\boldsymbol{x})$ monotonically decreases over time rules out the possibility of closed trajectories, given that they could only exist on level cures of V. Thus the use of Lyapunov functions is also an effective means for the preclusion of limit cycles.

Finally, it is possible to show that the existence of a Lyapunov function is intrinsically related to the stability properties of the equilibrium point as stated in the next theorem for the case that the origin is exponentially stable.

Theorem 1.6

Let $\boldsymbol{x} = \boldsymbol{0}$ be exponentially stable in $\mathscr{D} \subseteq \mathbb{R}^n$. Then there exists a Lyapunov function $V : \mathscr{D} \to \mathbb{R}$, $V(\boldsymbol{x}) > 0$ and a constant $\gamma > 0$ such that (1.13b) holds true.

Proof. By assumption there are constant a, λ such that $\|\boldsymbol{x}(t)\| \leq a\|\boldsymbol{x}_0\| e^{-\lambda t}$. Consider the functional

$$V(\boldsymbol{x}(t)) = \int_0^\infty \|\boldsymbol{x}(t+\tau)\|^2 d\tau.$$

It holds that

$$V(\boldsymbol{x}(t)) = 0, \quad \Leftrightarrow \|\boldsymbol{x}(t+\tau)\| = 0, \quad \forall \tau \geq 0,$$

showing that V is positive definite. On the other hand, in virtue of the exponential stability of the origin it holds that

$$V(\boldsymbol{x}(t)) \leq \int_0^\infty a^2\|\boldsymbol{x}(t)\|^2 e^{-2\lambda\tau} d\tau = \frac{a^2\|\boldsymbol{x}(t)\|^2}{2\lambda}$$

showing that for finite $\|\boldsymbol{x}_0\|$ the function $V(\boldsymbol{x})$ is quadratically bounded from above by the norm of $\boldsymbol{x}(t)$. Consider the rate of change of V at time t evaluated at the point $\boldsymbol{x}(t)$ given by

$$\begin{aligned}
\frac{dV}{dt}(\boldsymbol{x}(t)) &= \lim_{\tau\to 0^+} \frac{1}{\tau}\left(\int_\tau^\infty \|\boldsymbol{x}(t+s)\|^2 ds - \int_0^\infty \|\boldsymbol{x}(t+s)\|^2 ds\right) \\
&= -\lim_{\tau\to 0^+} \frac{1}{\tau}\int_0^\tau \|\boldsymbol{x}(t+s)\|^2 ds \\
&= -\|\boldsymbol{x}(t)\|^2 \\
&\leq -\frac{2\lambda}{a^2} V(\boldsymbol{x}(t))
\end{aligned}$$

showing that inequality (1.13b) holds with $\gamma = \frac{2\lambda}{a^2}$. □

This last results shows that the asymptotical and exponential stability of equilibria is intrinsically related to the existence of Lyapunov functions, in the sense that their existence implies stability properties and vice versa, the stability properties imply their existence. Thinking about the origins of the Lyapunov theory, stemming from potential and kinetic energies and interchanges between them allows to see that there are actually very fundamental relations in terms of simple scalar (energy) functions describing the complete behavior of potentially rather complex nonlinear systems.

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Lyapunov-based design techniques

2

2.1 Integrator backstepping

Consider the following system

$$\dot{x}_1 = f_1(x_1) + g(x_1)x_2, \qquad x_1(0) = x_{10} \tag{2.1a}$$
$$\dot{x}_2 = u, \qquad x_2(0) = x_{20}, \tag{2.1b}$$

with smooth vector fields $f_1(x_1)$ and $g(x_1) \neq 0$, $\forall x_1 \in D_1 \subseteq \mathbb{R}$. Classical examples for such dynamics are motivated for systems where the actual actuator dynamics have to be taken into account and can be summarized in form of an integrator (e.g. a pump with supplied electric current). In the following a way to exploit the particular structure of this system dynamics for stabilization of the origin by state-feedback is discussed. Consider in a first, auxiliary step the state x_2 as *virtual* control input. By the condition that $g(x_1) \neq 0$ in D_1 it follows that a simple (linearizing) control can be assigned of the form

$$x_2 = \frac{v - f(x_1)}{g(x_1)} = \mu(x_1)$$

with closed-loop dynamics (for x_1) given by

$$\dot{x}_1 = v, \quad x_1(0) = x_{10}.$$

A desired behavior for x_1 can be introduced by adequately choosing v, e.g. as

$$v = -kx_1$$

to obtain exponential convergence with rate k. Another, more general approach could be to consider the Lyapunov function

$$V_1(x_1) = \frac{1}{2}x_1^2, \quad \frac{dV_1(x_1)}{dt} = x_1 v \overset{!}{=} -Q_1(x_1) \tag{2.2}$$

for some desired $Q_1(x_1) = x_1 q_1(x_1) > 0$ what is achieved using

$$\mu(x_1) = \frac{-q_1(x_1) - f(x_1)}{g(x_1)}. \tag{2.3}$$

Choosing $W_1(x_1) = kx_1^2$, the aforementioned control $v = -kx_1$ is recovered. So, consider for the moment that v is chosen such that (2.2) holds. Clearly, this control is not implementable, as it was assumed that x_2 would be the control input. Thus, actually the difference betweeen x_2 and $\mu(x_1)$ has to be considered:

$$z = x_2 - \mu(x_1), \quad \dot{z} = u - \frac{\partial \mu(x_1)}{\partial x_1}\Big(f_1(x_1) + g(x_1)\underbrace{[z + \mu(x_1)]}_{=x_2}\Big). \tag{2.4}$$

Introducing the Lyapunov function candidate

$$V_z(z) = \frac{1}{2}z^2 \tag{2.5}$$

it follows that

$$\frac{dV_z(z)}{dt} = z\left(u - \frac{\partial\mu(x_1)}{\partial x_1}\Big(f_1(x_1) + g(x_1)[z+\mu(x_1)]\Big)\right).$$

Clearly, one can use this equation to find u in dependence of z and x_1 so that $\frac{dV_z(z)}{dt} < 0$. Nevertheless, this would neglect the dynamics of x_1 for the transient during which $x_2 \neq \mu(x_1)$. Thus, consider the system dynamics in (x_1, z) coordinates

$$\dot{x}_1 = f_1(x_1) + g(x_1)[z+\mu(x_1)], \qquad x_1(0) = x_{10} \tag{2.6a}$$

$$\dot{z} = u - \frac{\partial\mu(x_1)}{\partial x_1}\Big(f_1(x_1) + g(x_1)[z+\mu(x_1)]\Big), \qquad z(0) = z_0 \tag{2.6b}$$

and the joint Lyapunov function candidate

$$W(x_1, z) = V_1(x_1) + V_z(z). \tag{2.7}$$

The rate of change of $W(x_1, z)$ over time is given by

$$\begin{aligned}\frac{dW(x_1,z)}{dt} &= x_1 f_1(x_1) + x_1 g(x_1)[z+\mu(x_1)] + z\left(u - \frac{\partial\mu(x_1)}{\partial x_1}\Big(f_1(x_1) + g(x_1)[z+\mu(x_1)]\Big)\right)\\ &= -Q_1(x_1) + z\left(x_1 g(x_1) + u - \frac{\partial\mu(x_1)}{\partial x_1}\Big(f_1(x_1) + g(x_1)[z+\mu(x_1)]\Big)\right)\end{aligned}$$

Thus, to achieve the condition

$$\frac{dW(x_1,z)}{dt} \overset{!}{=} -Q_1(x_1) - Q_2(z) < 0$$

for some $Q_2(z) = zq_2(z)$ so that $Q_2(z) > 0$ it is sufficient to choose the control input u as

$$u = -q_2(z) - x_1 g_1(x_1) + \frac{\partial\mu(x_1)}{\partial x_1}\Big(f_1(x_1) + g(x_1)[z+\mu(x_1)]\Big) = \varpi(x_1, z) \tag{2.8}$$

with $\mu(x_1)$ given in (2.3). In terms of (x_1, x_2) this stabilizing controller can be written as

$$u = \alpha(x_1, x_2) := \varphi(x_1, z)\Big|_{z = x_2 - \mu(x_1)}.$$

This approach is known as integrator backstepping and can be summarized in the following two steps:

(i) Design x_2 as auxiliary (*virtual*) input variable to obtain the relationship $x_2 = \mu(x_1)$ for which asymptotic (or exponential) stability is ensured

(ii) Design the *actual* controller input $u = \varpi(x_1, x_2)$ so that the difference $z = x_2 - \mu(x_1)$ asymptotically (or exponentially) converges to zero, taking into account the (possibly open-loop unstable) of x_1 during the transient for which $x_2 \neq \mu(x_1)$.

This is put in the Lyapunov-framework in the way discussed above and can directly be generalized to the case where $\boldsymbol{x}_1$ is a vector in $\mathbb{R}^n$ and for the case that an integrator chain of n integrators separate the dynamics of $\boldsymbol{x}_1$ and the input u (so that the relative degree is at least n).

Theorem 2.1

Consider the system

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{f}(\boldsymbol{x}_1) + \boldsymbol{g}(\boldsymbol{x}_1)x_2, \qquad x_1(0) = x_{10}$$

$$\dot{x}_2 = u, \qquad x_2(0) = x_{20}$$

with $\boldsymbol{x}_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}$ and smooth vector fields $\boldsymbol{f}, \boldsymbol{g}$ with $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$. Let $\mu(x_1)$ be a continuously differentiable function with $\mu(0) = 0$ and $V_1(\boldsymbol{x}_1)$ a differentiable, positive definite (radially unbounded) function so that

$$\frac{\partial V_1(\boldsymbol{x}_1)}{\partial \boldsymbol{x}_1}\big(\boldsymbol{f}(\boldsymbol{x}_1) + \boldsymbol{g}(\boldsymbol{x}_1)\mu(\boldsymbol{x}_1)\big) \le -Q_1(x_1) \le 0. \tag{2.9}$$

Then the following holds true:

(i) If $Q(x_1) > 0$ then the feedback

$$u = \alpha(\boldsymbol{x}_1, x_2) = \frac{\partial \mu(\boldsymbol{x}_1)}{\partial \boldsymbol{x}_1}\Big(\boldsymbol{f}(\boldsymbol{x}_1) + \boldsymbol{g}(\boldsymbol{x}_1)x_2\Big) - \frac{\partial V_1(\boldsymbol{x}_1)}{\partial \boldsymbol{x}_1}\boldsymbol{g}(\boldsymbol{x}_1) - k(x_2 - \mu(x_1)) \tag{2.10}$$

with $k > 0$ asymptotically stabilizes the origin $[\boldsymbol{x}_1^T, x_2]^T = \mathbf{0}^T$ and

$$W(\boldsymbol{x}_1, x_2) = V_1(\boldsymbol{x}_1) + \frac{1}{2}\big(x_2 - \mu(x_1)\big)^2$$

is an associated Lyapunov function.

(ii) If $Q_1(\boldsymbol{x}_1) \ge 0$ then for the closed-loop system with the feedback control (2.10) the trajectories converge into the largest positively invariant subset of

$$\mathscr{W} = \left\{ \begin{bmatrix} \boldsymbol{x}_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^{n+1} \,|\, Q_1(\boldsymbol{x}_1) = 0, x_2 = \mu(\boldsymbol{x}_1) \right\}. \tag{2.11}$$

Proof. (i) According to the assumptions it holds that

$$\frac{dW(\boldsymbol{x}_1, x_2)}{dt} = \frac{\partial V_1(\boldsymbol{x}_1)}{\partial \boldsymbol{x}_1}\big(\boldsymbol{f}(\boldsymbol{x}_1) + \boldsymbol{g}(\boldsymbol{x}_1)[\mu(\boldsymbol{x}_1) + x_2 - \mu(x_1)]\big) + \\ + \big(x_2 - \mu(x_1)\big)\left(u - \frac{\partial \mu(\boldsymbol{x}_1)}{\partial \boldsymbol{x}_1}\right)\big(\boldsymbol{f}(\boldsymbol{x}_1) + \boldsymbol{g}(\boldsymbol{x}_1)x_2\big)$$

and taking into account (2.9) and (2.10) it follows that

$$\frac{dW(\boldsymbol{x}_1, x_2)}{dt} = -Q_1(\boldsymbol{x}_1) - k(x_2 - \mu(\boldsymbol{x}_1))^2 < 0. \tag{2.12}$$

Given that $Q_1(\boldsymbol{x}_1) > 0$ by assumption, the asymptotic stability follows from Theorem 1.4.

(ii) For $Q_1(\boldsymbol{x}_1) \geq 0$ it follows from Theorem 1.2 that the trajectories converge into the largest positively invariant subset of

$$\mathscr{W}_0 = \left\{ \begin{bmatrix} \boldsymbol{x}_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \frac{dW(\boldsymbol{x}_1, x_2)}{dt} = 0 \right\}.$$

According to (2.12) this set is identical with (2.11).

□

Beyond this result, one can directly consider chains of integrators, like in the system

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{f}(\boldsymbol{x}_1) + \boldsymbol{g}(\boldsymbol{x}_1)x_2, & x_1(0) &= x_{10} \\ \dot{x}_2 &= x_3, & x_2(0) &= x_{20} \\ &\vdots \\ \dot{x}_n &= u, \quad x_n(0) = x_{n0} \end{aligned}$$

or the more general set-up where

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2), & \boldsymbol{x}_1(0) &= \boldsymbol{x}_{10} \\ \dot{\boldsymbol{x}}_2 &= \boldsymbol{f}_2(\boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{u}, & \boldsymbol{x}_2(0) &= \boldsymbol{x}_{20} \end{aligned}$$

under the assumption that vector field $\boldsymbol{\alpha}(\boldsymbol{x}_1)$ and a Lyapunov function $V(\boldsymbol{x}_1)$ is known which proofs the asymptotic stability of the system

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{\alpha}(\boldsymbol{x}_1)), \quad \boldsymbol{x}_1(0) = \boldsymbol{x}_{10}.$$

For more information, the interested reader is referred to the literature [SJK97; Kug17].

2.2 Dissipativity and passivity-based control

Consider the SISO control system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{2.13a}$$

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \tag{2.13b}$$

with state $\boldsymbol{x} \in \mathbb{R}^n$, input $\boldsymbol{u} \in \mathbb{R}^p$ and output $\boldsymbol{y} \in \mathbb{R}^m$ and smooth vector fields $\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{g}_i(\boldsymbol{x})$, $i = 1, \ldots, p$, $G(\boldsymbol{x}) = [\boldsymbol{g}_1(\boldsymbol{x}), \ldots, \boldsymbol{g}_p(\boldsymbol{x})]$ and differentiable output map $\boldsymbol{h}(\boldsymbol{x})$.

The notion of passivity in systems theory is motivated by the notion of passivity in electrical engineering, where this concept refers to an electrical circuit in which the electric power consumption $P = UI$ is always positive, in the understanding that this implies that the circuit dos not produce energy by itself and the net flow of energy is always into the circuit. This has been extended to a general set-up for (open) dynamical (control) systems in a state space framework [Kal64; Kal63; Moy74; HM76; BIW91; Wil72a; Wil72b; HM80; Bro+07]. In order to state the notions of dissipativity and passivity the concept of internal (energy) storage is used, generalizing the concept of energy in terms of the system state $\boldsymbol{x}$. The second concept which is employed is a generalization of the power supply to a system represented in form of a general supply rate $\omega(\boldsymbol{u}, \boldsymbol{y})$.

With these concepts, the notions of dissipativity and passivity can be defined in the following way.

Definition 2.1

(i) The system (2.13) is called dissipative with respect to the supply rate $\omega(\boldsymbol{u},\boldsymbol{y})$ if there exists a (positive semi-definite) storage function $S(\boldsymbol{x}) \geq 0$ so that

$$S(\boldsymbol{x}(t)) - S(\boldsymbol{x}_0) \leq \omega(\boldsymbol{u},\boldsymbol{y}), \quad \forall \boldsymbol{x}_0.$$

If S is differentiable, this relation can be written equivalently as

$$\frac{dS(\boldsymbol{x})}{dt} = \frac{\partial S(\boldsymbol{x})}{\partial \boldsymbol{x}}\big(\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}\big) \leq \omega(\boldsymbol{u},\boldsymbol{y}).$$

(ii) System (2.13) with $m = p$ is called passive if it is dissipative with respect to the supply rate $\omega(\boldsymbol{u},\boldsymbol{y}) = \boldsymbol{u}^T\boldsymbol{y}$.

(iii) A static map $\boldsymbol{y} = \boldsymbol{\varphi}(\boldsymbol{u})$ with $m = p$ is called passive if it holds that $\boldsymbol{u}^T\boldsymbol{y} = \boldsymbol{u}^T\boldsymbol{\varphi}(\boldsymbol{u}) \geq 0$.

Example 2.1. *A very simple example for a passive electrical element is a resistance R for which it holds by Ohm's law that $u_R = Ri_R$ and thus, taking i_R as the input and u_R as the output*

$$u_R = \varphi(i_R) = Ri_R, \quad \omega(i_R, u_R) = i_R u_R = i_R R i_R = R i_R^2 > 0.$$

Example 2.2. *A more interesting example with an internal memory is given by the RLC circuit with a current source (see Fig. 2.1). For the voltage over the capacitance it holds that*

$$\frac{du_C}{dt} = -\frac{1}{RC}u_C + \frac{1}{C}i_q, \quad u_C(0) = u_{C0}$$
$$y = u_C$$

so that taking the current source as input, i.e. $u = i_q$ and the electrical energy of the capacitance as storage function, i.e. $S = \frac{1}{2}Cu_C^2 > 0$ it follows that

$$\frac{dS(u_C)}{dt} = -\frac{1}{R}u_C^2 + i_q u_C \leq i_q u_C = uy$$

implying the passivity of this simple electrical circuit.

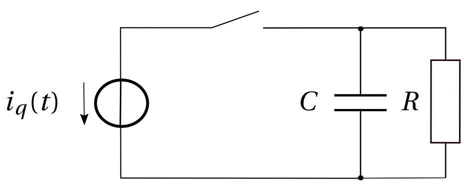

Figure 2.1: RC-circuit in example 2.1.

The main idea in passivity-based control is to use the storage function as Lyapunov-function candidate and exploit the structure of the supply rate for (output- or state-) feedback control design. Recalling the fundamental results from Lyapunov stability theory presented in Chapter 1, it is clear that as long as it is possible to find a control law in such a way that the rate of change in the storage is negative semi-definite, the stability follows if $S(\boldsymbol{x}) > 0$. In the case that even negative definiteness can be achieved the asymptotic (or exponential) stability of the origin can be ensured.

To analize this feature further, note that in example 2.1 of the RC circuit an interesting additional property can be observed. To show the passivity of the system, in the final step the (negative definite) term $-\frac{1}{R}u_C^2 < 0$ has been neglected. Nevertheless, from the point of view of Lyapunov theory this term already ensures that for $u = 0$ one has

$$\frac{dS(u_C)}{dt} = -\frac{1}{R}u_C^2 < 0$$

implying the (exponential) asymptotic stability of the solution $u_C = 0$. There are a couple of different notions in the theory of dissipative and passive systems to handle such properties (see e.g. [SJK97; Kha96; Bro+07]). The one employed here is defined next.

Definition 2.2

System (2.13) is called strictly state dissipative with respect to the supply rate $\omega(\boldsymbol{u}, \boldsymbol{y})$ if there exists a non-negative (i.e., positive semi-definite) storage function $S(\boldsymbol{x}) \geq 0$ and a constant $\kappa > 0$ such that

$$\frac{dS(\boldsymbol{x})}{dt} \leq \omega(\boldsymbol{u}, \boldsymbol{y}) - \kappa\|\boldsymbol{x}\|^2.$$

It is called strictly (state) passive if it is strictly (state) dissipative with respect to the supply rate $\omega(\boldsymbol{u}, \boldsymbol{y}) = \boldsymbol{u}^T \boldsymbol{y}$.

In particular, when a system is strictly state passive with a positive definite storage function $S(\boldsymbol{x}) > 0$ it can be (exponentially) asymptotically stabilized using the simply linear feedback control law $\boldsymbol{u} = -K\boldsymbol{y}$ with $K \geq 0$, given that then

$$\frac{dS(\boldsymbol{x})}{dt} \leq -\kappa\|\boldsymbol{x}\|^2 + \boldsymbol{u}^T\boldsymbol{y} \overset{\boldsymbol{u}=-K\boldsymbol{y}}{=} -\kappa\|\boldsymbol{x}\|^2 - \boldsymbol{y}^T K\boldsymbol{y} \leq -\kappa\|\boldsymbol{x}\|^2 < 0.$$

This reasoning allready explains why passive systems have quite useful properties for control design, besides their importance in electrical engineering. In order to enable similar results for the more general class of dissipative systems typically some further structural constraints are introduced for the supply rate. For this purpose the studies are focussed on the case of quadratic supply rates of the form

$$\omega(\boldsymbol{u}, \boldsymbol{y}) = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{y} \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{y} \end{bmatrix}$$

and the related dissipativity property is then called (Q, S, R)-dissipativity. This opens a wide field of oportunities which to cover goes beyond the scope of these notes and the interested reader is referred to the related literature (see e.g. [Wil72b; Bro+07]).

Motivated by these initial considerations, in the following the focus is put on the class of passive systems. Consider again the case of a passive system with a positive definite storage function $S(\boldsymbol{x}) > 0$ and the feedback control

$$\boldsymbol{u} = -K\boldsymbol{y}, \quad K > 0.$$

Accordingly, in virtue of

$$\frac{dS(\boldsymbol{x})}{dt} \leq -\boldsymbol{y}^T K\boldsymbol{y} \leq 0$$

it follows from Krasovskiy-LaSalles invariance principle (see Theorem 1.2) that the state converges into the largest positively invariant subset of

$$\mathscr{Y}_0 = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \equiv \boldsymbol{0}\}. \tag{2.14}$$

The positive invariance goes at hand with the condition that $\boldsymbol{y} \equiv \boldsymbol{0}$, or equivalently, that $\boldsymbol{y}^{(k)} = 0$ for all $0 \leq k \in \mathbb{N}_0$. Using the notion of the Lie-derivative

$$L_{\boldsymbol{f}} h_i(\boldsymbol{x}) = \frac{\partial h_i(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}), \quad L_{\boldsymbol{f}}^k h_i(\boldsymbol{x}) = \frac{\partial L_{\boldsymbol{f}}^{k-1} h_i(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}), \quad L_{\boldsymbol{f}}^0 h_i(\boldsymbol{x}) = h_i(\boldsymbol{x})$$

for $i = 1, \ldots, m$, this restriction implies that

$$\begin{aligned} y_i &= h_i(\boldsymbol{x}) = 0 \\ \dot{y}_i &= L_{\boldsymbol{f}} h_i(\boldsymbol{x}) = 0 \\ y_i^{(2)} &= L_{\boldsymbol{f}}^2 h_i(\boldsymbol{x}) = 0 \\ &\vdots \\ y_i^{(n)} &= L_{\boldsymbol{f}}^n h_i(\boldsymbol{x}) = 0 \end{aligned}$$

needs to hold true for all $i = 1, \ldots, m$. In vector form this implies that

$$\mathscr{O}(\boldsymbol{x}) = \begin{bmatrix} h_i(\boldsymbol{x}) \\ L_{\boldsymbol{f}} h_i(\boldsymbol{x}) \\ \vdots \\ L_{\boldsymbol{f}}^n h_m(\boldsymbol{x}) \end{bmatrix} = \boldsymbol{0}. \tag{2.15}$$

The map $\mathscr{O}(\boldsymbol{x})$ is the nonlinear observability map[1] [Isi95; NS90], so that in case that the system is completely observable –in the sense that this map is everywhere uniquely invertible – it turns out that only the zero vector $\boldsymbol{x} = \boldsymbol{0}$ is a solution of (2.15) and thus the asymptotic stability of $\boldsymbol{x} = \boldsymbol{0}$ follows.

In linear systems, if the map can be inverted along one trajectory, it can be inverted along any trajectory[2]. In nonlinear systems this is not true and it is worth introducing a new concept which corresponds to the invertibility along the solution (and thus uniqueness of this solution) for which $\boldsymbol{y} \equiv 0$.

Definition 2.3

The system (2.13) is called

(i) *zero-state observable* if $\boldsymbol{y} \equiv \boldsymbol{0}$ implies $\boldsymbol{x} = \boldsymbol{0}$, i.e. if the solution of $\mathscr{O}(\boldsymbol{x}) = \boldsymbol{0}$ is uniquely given by $\boldsymbol{x} = \boldsymbol{0}$.

(ii) *zero-state detectable* if $\boldsymbol{y} \equiv \boldsymbol{0}$ implies $\boldsymbol{x} \to \boldsymbol{0}$.

[1] It can be quickly shown, that for a linear system this map corresponds to the Kalman observability map $\mathscr{K}_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$.

[2] For linear systems the map (2.15) can actually be written as the Kalman observability matrix $\mathscr{K}_o$ times the state vector $\boldsymbol{x}$.

From the definition it should be clear, that zero-state observability implies zero-state detectability but not *vice versa*. Note that if the system is not zero-state observable but zero-state detectable, then the map $\mathscr{O}(\boldsymbol{x})$ is not invertible, but all solutions $\boldsymbol{x}(t)$ which are mapped by $\mathscr{O}$ to the zero vector $\mathbf{0}$ converge asymptotically to zero. This implies that by Krasovskiy-LaSalles invariance principle all trajectories converge into a set where the origin is the unique attractor.

To simplify the notation and focus on the main concepts let us restrict the considered class of systems to the SISO case

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + g(\boldsymbol{x})u, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{2.16a}$$

$$y = h(\boldsymbol{x}). \tag{2.16b}$$

Looking on the constraint $y \equiv 0$ in the definition of *zero-state detectability* the notion of the *zero dynamics* comes into play. The zero dynamics is given by (2.16) with the constraint $y \equiv 0$ and u chosen so that this constraint holds true. To make this point clear, note that if the relative degree [Isi95; SJK97] of (2.16) is equal to one at $\boldsymbol{x} = 0$, i.e. there exists a neighborhood $\mathscr{N}_0$ of $\boldsymbol{x} = \mathbf{0}$ so that

$$L_{\boldsymbol{g}} h(\boldsymbol{x}) \neq 0, \quad \forall\, \boldsymbol{x} \in \mathscr{N}_0,$$

and a diffeomorphic[3] state transformation $\boldsymbol{\Psi} : \mathscr{N}_0 \to \mathbb{R}^n$ of the form

$$\begin{bmatrix} z \\ \boldsymbol{\zeta} \end{bmatrix} = \boldsymbol{\Psi}(\boldsymbol{x}) = \begin{bmatrix} h(\boldsymbol{x}) \\ \boldsymbol{\Phi}(\boldsymbol{x}) \end{bmatrix}, \quad z \in \mathbb{R}, \quad \boldsymbol{\zeta} \in \mathbb{R}^{n-1}. \tag{2.17}$$

The dynamics in the new coordinates reads

$$\dot{z} = L_{\boldsymbol{f}} h(\boldsymbol{x}) + L_{\boldsymbol{g}} h(\boldsymbol{x}) u, \qquad z(0) = z_0 \tag{2.18a}$$

$$\dot{\boldsymbol{\zeta}} = \boldsymbol{\varphi}(z, \boldsymbol{\zeta}, u), \qquad \boldsymbol{\zeta}(0) = \boldsymbol{\zeta}_0 \tag{2.18b}$$

$$y = z. \tag{2.18c}$$

Actually, it is possible to choose the map $\boldsymbol{\Phi}(\boldsymbol{x})$ so that the vector field $\boldsymbol{\varphi}$ does not depend on the input u, but this does not make a difference at this stage. The zero dynamics is given by

$$\dot{\boldsymbol{\zeta}} = \boldsymbol{\varphi}(0, \boldsymbol{\zeta}, u) =: \boldsymbol{\varphi}_0(\boldsymbol{\zeta}), \qquad \boldsymbol{\zeta}(0) = \boldsymbol{\zeta}_0 \tag{2.19a}$$

$$u = -\left.\frac{L_{\boldsymbol{f}} h(\boldsymbol{x})}{L_{\boldsymbol{g}} h(\boldsymbol{x})}\right|_{\boldsymbol{x} = \boldsymbol{\Psi}^{-1}\left(\begin{bmatrix} 0 \\ \boldsymbol{\zeta} \end{bmatrix}\right)} \tag{2.19b}$$

With these notions at hand the following result is a direct consequence of the passivity property and the application of Lyapunov's direct method.

Theorem 2.2

Let (2.16) be passive with positive definite storage function $S(\boldsymbol{x}) > 0$. Then the following holds true:

(i) The zero dynamics (2.19) is Lyapunov stable.

[3] A map is a diffeomorphism if it is continuously differentiable and invertible, with continuously differentiable inverse. Such maps conserve geometric and topological properties in state space and are frequently used in the theory of dynamical systems, in particular for control design purposes.

(ii) If (2.16) is zero-state observable then the output-feedback control $u = -ky$ asymptotically stabilizes the origin $\boldsymbol{x} = \boldsymbol{0}$.

Proof. (i) Under the assumptions of the proposition, the storage function $S(\boldsymbol{x}) > 0$ is a Lyapunov function. From the passivity property it follows that for any state trajectory which is a solution of the zero dynamics (2.19) it holds that $\frac{dS(\boldsymbol{x})}{dt} \leq yu = 0$ implying Lyapunov stability in virtue of Theorem 1.1.

(ii) From Theorem 1.2 it follows that with $u = -ky$ the state trajectories $\boldsymbol{x}(t)$ converge into the largest positively invariant subset $\mathcal{M} \subseteq \mathcal{Y}_0$, with $\mathcal{Y}_0$ defined in (2.14). By the zero-state observability assumption this subset is given by $\mathcal{M} = \{\boldsymbol{0}\}$.

□

In the following the question is addressed if it is possible to passivate a system using feedback control. For this purpose, recall the state transformation (2.17) along with the dynamics in the new coordinates (2.18). According to the relative degree one property, it follows that using the control

$$u = \frac{v - L_{\boldsymbol{f}} h(\boldsymbol{x})}{L_{\boldsymbol{g}} h(\boldsymbol{x})} \tag{2.20}$$

and introducing the positive semi-definite storage function $S(\boldsymbol{x}) = \frac{1}{2}z^2$, the system is passive with respect to the new input v and it holds that

$$\frac{dS(\boldsymbol{x})}{dt} = zv = yv.$$

From this relation alone, it is nevertheless not possible to conclude about the zero dynamics, given that here $S(\boldsymbol{x}) \geq 0$ is only positive semi-definite. The theory of Lyapunov has been extended to positive semi-definite Lyapunov functions, and is related to the concept of *conditional stability*, but treating this subject goes beyond the scope of the present notes. The reader can explore this interesting subject e.g. in the seminal work [SJK97] or the related literature. For the purpose at hand, we focus on positive definite storage functions $S(\boldsymbol{x}) > 0$.

The following concept further characterizes systems in dependence of the properties of the zero dynamics (2.19) [BIW91; SJK97].

Definition 2.4

Let $L_{\boldsymbol{g}} h(\boldsymbol{0}) \neq 0$. Then (2.16) is said to be

- *minimum phase* if $\boldsymbol{\zeta} = \boldsymbol{0}$ is a locally asymptotically stable equilibrium point of the zero dynamics (2.19)
- *weakly minimum phase* if there exists a positive definite Lyapunov function $V_0(\boldsymbol{\zeta}) > 0$ defined in a neighborhood $\mathcal{N}_{\boldsymbol{\zeta},\boldsymbol{0}}$ of $\boldsymbol{\zeta} = \boldsymbol{0}$ that is at least 2 times continuously differentiable and satisfies $L_{\boldsymbol{\varphi}_0} V_0(\boldsymbol{\zeta}) \leq 0$ for all $\boldsymbol{\zeta} \in \mathcal{N}_{\boldsymbol{\zeta},\boldsymbol{0}}$.

Note that the weakly minimum phase property implies the Lyapunov stability of $\boldsymbol{\zeta} = \boldsymbol{0}$ (see also the discussion in [BIW91]).

Having these concepts at hand, the following result is stated without its proof which can be found in the related literature [BIW91; SJK97] (and directly extends to the MIMO case).

Theorem 2.3

The system (2.16) is locally feedback equivalent to a passive system (i.e. there exists a feedback such that the closed-loop system is passive) with a C^2 positive definite storage function $S(\boldsymbol{x}) > 0$ if and only if it has relative degree $r = 1$ at $\boldsymbol{x} = \boldsymbol{0}$ and is weakly minimum phase.

Accordingly, a system which is feedback eequivalent to a passive system can be stabilized by the feedback control

$$u = \frac{-ky - L_{\boldsymbol{f}} h(\boldsymbol{x})}{L_{\boldsymbol{g}} h(\boldsymbol{x})}.$$

A direct extension of this result holds for the case that the system is minimum phase. In this case the origin can be asymptotically stabilized using the above control law.

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